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Statistical Analysis of Effective Singular Values in Matrix Rank Determination

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ABSTRACT

A major problem in using SVD as a tool in determining the effective rank of a perturbed matrix, is that of distinguishing between significantly small and insignificantly large singular values. In this paper, we derive confidence regions for the perturbed singular values of matrices with noisy observation data. The analysis is based on the theories of perturbations of singular values and statistical significance test. Threshold bounds for perturbation due to finite precision and i.i.d. random models are evaluated. In random models, the threshold bounds depend on the dimension of the matrix, the noise variance, and a predefined statistical level of significance. Results applied to the problem of determining the effective order of a linear AR system from the approximate rank of a sample autocorrelation matrix are considered. Various numerical examples illustrating the usefulness of these bounds are given.

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I. Introduction

One of the most stable and computationally effective algorithms in the theory of matrix algebra is the Singular Value Decomposition (SVD). SVD distinguishes itself from the other decomposition algorithms because it is particularly effective in the presence of round-off errors or noisy data.

In the theory of SVD, [1;p.318], any m x n real valued matrix A of rank r, where $r \leq \min(m,n)$, can be decomposed as $A = U \Sigma V^T$, where U is an m x n matrix with $U^TU = I$, V is an orthogonal n x n matrix, and Σ is an n x n diagonal matrix. The diagonal elements $\alpha_1, \alpha_2, \ldots \alpha_n$, of Σ can be arranged in a non-increasing order and are called the singular values of A (SV). All the elements are nonnegative and exactly r of them are strictly positive. Specifically,

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r > \alpha_{r+1} = \ldots = \alpha_n = 0. \tag{1.1}$$

Because SVD provides an effective and computationally efficient method of determining the effective rank of a matrix, in recent years it has been extensively applied in a number of least squares, spectral estimation and system identification problems [1-15].

In theory, the rank of A can be determined by counting the number of positive SV's . In practice the actual observed matrix B consists of the original matrix A perturbed by round-off errors due to finite precision numerical operations and (or) random noises.

The simplest, yet meaningful, procedure is to use an additive perturbation model given by

(1.2) $B=A+E.$

While the original m x n data matrix A is assumed to have rank $r \le (m,n)$, the perturbation matrix E is assumed to have small values but is of full rank. Then the SV Decomposition of B will give a

(1.3) $\Sigma = \mathrm{diag}\ (\beta_1,\ldots,\beta_n),$

where $\beta_1 \ge \beta_2 \ge \ldots \ge \beta_r \ge \beta_{r+1} \ge \ldots \ge \beta_n \ge 0$, and $\{\beta_{r+1},\beta_{r+2},\ldots,\beta_n\}$, will usually be small but not necessarily zero. Since β_r may also be small, the difficult problem we face now in rank determination is to determine what is meant by the number of significant (large) SV's or equivalently the number of insignificant (small) SV's.

In order to illustrate these points, consider the following 3 x 3 matrix A.

Example 1.

(1.4) $$A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 5 & 6 \\ 2 & 1 & 3 \end{bmatrix}.$$

Since the first two columns are linearly independent and the third column is the sum of the first two columns, the matrix is clearly of rank 2. Indeed, the SV's of A are 10.342, 2.653, and 2.363 x E-17 (as evaluated on a double precision computer with numerical precision of less than 16 digits). Thus, from these SV's, we can easily determine the rank 2 status of A.

Now, due to noisy observation, instead of A, suppose we have a perturbed matrix B given by

$$(1.5) \qquad B = \begin{bmatrix} 2.998 & 1.999 & 5.001 \\ 1.001 & 4.982 & 5.998 \\ 2.001 & 1.001 & 2.990 \end{bmatrix}.$$

Clearly, the third column is no longer the sum of the first columns and indeed is not obviously a linear combination of the first two columns. Direct evaluation of the determinant of B yields a value of -0.176. This indicates B is not singular and is of rank 3. Similarly, by using Gaussian elimination, B can be reduced to an upper triangular matrix B_G given by

$$(1.6) \quad B_G = \begin{bmatrix} 2.998 & 1.999 & 5.001 \\ 0 & -12.922 & -12.963 \\ 0 & 0 & 0.577 \end{bmatrix}.$$

Since the diagonals of B_G are clearly non-zero, then B_G and thus B are also of rank 3.

However, a SV Decomposition of B yields SV's of 10.331, 2.644, and 0.0064. Since the first two SV's are significantly larger than the relatively small third SV, we may be able to conclude that B may be the result of a noisy perturbation of a rank 2 matrix. In general, we want to have a precise method of rank determination by using the observed SV's and some information of the perturbations.

The determination of the effective rank $t \leq n$ of an observed matrix B by using its SV's has been considered previously based on various criteria.

(1.7) Criterion 1. $\beta_1 \geq \beta_2 \ldots \geq \beta_t > \delta_1 \geq \beta_{t+1} \geq \ldots \geq \beta_n$

(1.8) Criterion 2. $\beta_t/\beta_1 > \delta_2 > \beta_{t+1}/\beta_1$

(1.9) Criterion 3. $\beta_t >> \beta_{t+1}$

(1.10) Criterion 4. $\beta_{t+1}^2 + \beta_{t+2}^2 + \dots + \beta_n^2 < \delta_4$

(1.11) Criterion 5. $$\left[\frac{\beta_1^2 + \beta_2^2 + \dots + \beta_t^2}{\beta_1 + \beta_2^2 + \dots + \beta_n^2}\right]^{1/2} > \delta_5.$$

Criteria 1-5 have been considered in [10;p.176], [5;p.114], [5;p.117], [1;p.332] and [11], respectively. Intuitively, all these criteria appear reasonable and indeed may work in various cases. However, the threshold values of δ_2, δ_4, and δ_5 do not appear to be based on any explicit analytical expressions but are selected on an ad hoc basis. For a specific finite precision round-off model, an analytical expression for δ_1 was given in [10;p.176].

In Section II, we consider some preliminary matrix algebra definitions and bounds as well as some eigenvalue perturbation results summarized into two theorems. In Section III, we define the effective rank of a matrix based on that of criterion 1 and derive analytically the upper and lower bounds on δ_1. Bounds based on that of a finite precision model, an i.i.d. random model, and a column random model are considered. For the random perturbation models, these threshold bounds are derived and interpreted in terms of theory of statistical significance test. A detailed example shows that our bounds are tight and useful. In Section IV, we consider an application of this technique to the determination of the order of an AR system model.

II. Preliminary Results from Matrix Algebra

In this section, we collect together various inequality results from matrix algebra related to norm of vectors and matrices and the perturbation of singular values needed for the discussions in the next section. Basic concepts and detailed expositions on these matters can be found in the books of [1], [3], and [10].

Consider an m x n matrix A with real-valued elements $[a_{ij}]$ and singular values $\alpha_1 \geq \alpha_2 \geq \ldots \alpha_k \geq 0$ where $k \leq \min(\text{min},n)$. Then the Frobenious norm (F-norm) of A is defined as

$$(2.1) \qquad \|A\|_F = [\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2]^{1/2} = [\sum_{i=1}^{k} \alpha_i^2]^{1/2},$$

and the 2-norm of A is defined as

$$(2.2) \qquad \|A\|_2 = \operatorname*{Max}_{x \neq o} (\|Ax\|_2 / \|x\|_2) = \alpha_1,$$

where the 2-norm of $x=(x_1,\ldots,x_n)'$ is defined as

$$(2.3) \qquad \|x\|_2 = (x_1^2 + \ldots + x_n^2)^{1/2}.$$

The desired inequalities are stated in two theorems. The proof of Theorem 1 is given in Appendix A while the proof of Theorem 2 can be found in [3;p.25-26].

<u>Theorem 1.</u>

For any real-valued m x n matrix A with column vectors $a_1, a_2, \ldots, a_n$ and norms defined in (2.1), (2.2), and (2.3), the following inequalities are valid

$$(2.4) \quad \max |a_{ij}| \leq \max \|a_j\|_2 \leq \|A\|_2 \leq \|A\|_F \leq \sqrt{n} \max \|a_j\|_2 \leq \sqrt{mn} \max |a_{ij}| .$$

<u>Theorem 2.</u>

Let A, B, and E be m x n real-valued matrices with B=A+E. Denote their respective singular values by α_i, β_i and ϵ_i, i=1,2,...,k,

$k \le \min(m,n)$, each set labeled in nonincreasing order. Then

(2.5) $$|\beta_i - \alpha_i| \le \epsilon_1 = \|E\|_2, \quad i=1,2,\ldots,k.$$

III. Statistical Threshold Bounds

In this section, we shall use results of Section II to derive threshold bounds for effective matrix rank determination based on three models of perturbations in the observations. From Theorem 2, if A is an $m \times n$ matrix of rank r, where $r \leq \min(m,n)$, with singular values $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r > 0$, E is an $m \times n$ perturbation matrix with elements $[e_{ij}]$ and 2-norm $\|E\|_2 = \epsilon_1$, and $B = A+E$, with singular values $\beta_1 \geq \ldots \geq \beta_r \geq \beta_{r+1} \ldots \geq \beta_n$, then

(3.1) $|\beta_{r+1} - \alpha_{r+1}| = \beta_{r+1} \leq \epsilon_1 = \|E\|_2.$

Hence, if $\beta_r > \epsilon_1$, then

(3.2) $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_r > \epsilon_1 \geq \beta_{r+1} \geq \beta_{r+2} \geq \ldots \geq \beta_n.$

As a result of (3.2), we will use the following definition for the effective rank of B.

<u>Definition:</u> For any $m \times n$ matrix $B = A+E$, as defined above, the effective rank of B is defined to be r, when

(3.3) $\beta_r > \epsilon_1 \geq \beta_{r+1},$

where $1 \leq r \leq \min(m,n)$ and $\epsilon_1 = \|E\|_2$ is the 2-norm of E.

Clearly, the above definition of effective rank is based on criterion 1 in (1.7) with an explicit construction of $\delta_1 = \epsilon_1$. In light of this definition, a simple sufficient condition for the determination of the effective rank of B in terms of the singular values of A and E is considered in Theorem 3. The proof of Theorem 3 is given in Appendix B.

<u>Theorem 3.</u>

Let A, B, and E be $m \times n$ matrices as defined above; with the 2-norm of E denoted by ϵ_1. If $\alpha_r > 2\epsilon_1$, then $\beta_r > \epsilon_1 \geq \beta_{r+1}$ and B is said to have effective rank of r.

While Theorem 3 guarantees the determination of the effective rank of B if $\alpha_r > 2\epsilon_1$, in practice, we may not know the singular values $\{\alpha_i\}$ of A, as well as the elements of E are not necessarily fully known. In the rest of this section, we shall derive upper and lower bounds on ϵ_1 to be used in (3.3) for effective rank determination under various cases of interest.

A. Finite Precision Model

First, consider the finite precision case where the elements $[e_{ij}]$ of E satisfy

(3.4) $-\Delta/2 \le e_{ij} \le \Delta/2,\ 1 \le i \le m,\ 1 \le j \le n.$

Here Δ may be the step size of an A/D converter or may be related to the round-off errors in some computations. In general, Δ can be taken to be small and at least one of the inequalities in (3.4) is assumed to be attained for some e_{ij}. Let e_j denote the j-th column of E. Then

(3.5) $\max |e_{ij}| = \Delta/2,$

(3.6) $\max_j \|e_j\|_2 = \max\ [\,|e_{1j}|^2 \ldots + |e_{mj}|^2]^{1/2} \le \sqrt{m}\ \Delta/2.$

From Theorem 1 and (3.5)-(3.6), we obtain

(3.7) $\Delta/2 \le \|E\|_2 = \epsilon_1 \le \sqrt{mn}\ \Delta/2.$

Example 2.

Consider the 3x3 perturbed matrix B of (1.5) in Example 1. By comparison to the matrix A of (1.4), we have $\Delta/2 = 0.018$. Then (3.7) yields

(3.8) $0.018 \le \epsilon_1 \le 0.054.$

Since the SV's of B are {10.331, 2.644, 0.0064}, then $\beta_2 = 2.644 > 0.054 \ge \epsilon_1 \ge 0.018 > \beta_3 = 0.0064$ shows that the effective rank of B is predicted correctly as r=2.

B. I.I.D. Random Model

In many practical situations, the perturbation matrix E arises from noises in the measurement modeled by

(3.9) $b_{ij} = a_{ij} + e_{ij}$, $1 \le i \le m$, $1 \le j \le n$.

Thus, there are many possible random variables for characterizing $\{e_{ij}\}$. One of the simplest, yet quite meaningful models is to assume $\{e_{ij}\}$ to be i.i.d. random variables with a Gaussian density of zero mean and known variance σ^2. While the r.v., e_{ij}, can take values over the entire real line, we can define a finite region such that the r.v. is in this region with high probability. Specifically, denote $K = \{e_{ij}: -k \le e_{ij} \le k\}$ and $K' = \{e_{ij}: e_{ij} \le -k \text{ or } k > e_{ij}\}$. Since Prob (K) + Prob (K') = 1, a large Prob (K) is equivalent to a small Prob (K'). From the theory of statistical test [16;pp.334], if α denotes the level of significance of the test, then

(3.10) $\text{Prob } (K') = \text{Prob } [|e_{ij}| > k] \le \alpha$,

For i.i.d. Gaussian $\{e_{ij}\}$,

(3.11) $\text{Prob } (K') = 2 \text{ Prob } [e_{ij} > k] = 2(1 - \Phi(k/\sigma))$,

where $\Phi(\cdot)$ is the zero-mean and unit variance Gaussian probability distribution function

(3.12) $$\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} \exp -(t^2/2)dt.$$

From (3.10)-(3.11),

(3.13) $1 - \Phi(k/\sigma) = \alpha/2$

or

(3.14) $k = \sigma\, \Phi^{-1}(1 - (\alpha/2))$,

From (3.14) and the Gaussian probability distribution table [16;p.557], $\alpha = 0.01$ yields approximately $k_1 = 2.6\sigma$, while

$\alpha = 0.05$ yields approximately $k_2 = 2\sigma$. The operational significance of these results for any e_{ij} is that with 0.99 probability confidence we have

(3.15) $\max |e_{ij}| = k_1 = 2.6\sigma$,

and with 0.95 probability confidence, we have

(3.16) $\max |e_{ij}| = k_2 = 2\sigma$.

The maximization in (3.15) or (3.16) is with respect to the values of the realization of the r.v. $|e_{ij}|$ for any i or j. Thus by using a sufficiently large k, we will have high probability confidence that $|e_{ij}| \leq k$. By using the first and fifth inequalities of (2.4) in Theorem 1 and (3.14), we have

<u>First Bounds</u>

(3.17) $k \leq \epsilon_1 \leq \sqrt{mn}\, k$.

For large values of mn and σ, the lower and upper bounds in (3.17) may not be tight and indeed may become useless.

However, by using the second and fourth inequalities of (2.4) in Theorem 1, we can obtain tighter bounds. For any j, denote

(3.18) $S^2 = \|e_j\|_2^2 = \sum_{i=1}^{m} |e_{ij}|^2$.

Then S^2/σ^2 has a Chi-square distribution with m degrees of freedom [16;p.233-234]. Therefore, for a given level of significance α, a constant c (corresponding to k of (3.10).) can be found such that

(3.19) Prob $[S^2/\sigma^2 > c] \leq \alpha$.

Various values of c as a function of the level of significance α and the degrees of freedom m are tabulated [16;p.559] (corresponding to k of (3.14)). Then from Theorem 1 and (3.18)-(3.19), we have

Second Bounds

(3.20) $\sqrt{c}\,\sigma \le \epsilon_1 \le \sqrt{nc}\,\sigma$.

Finally, by using the second and third inequalities of (2.4) in Theorem 1, we obtain an even tighter bound on the right of ϵ_1. Let

(3.21) $S_F^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} |e_{ij}|^2$

Then as before, S_F^2/σ^2 has a Chi-square distribution of mn degrees of freedom. Then as in (3.19), for a given α, a constant c_F can be found such that

(3.22) Prob $[S_F^2/\sigma^2 > c_F] \le \alpha$.

By using Theorem 1 and (3.21)-(3.22), we have the

Third Bounds

(3.23) $\sqrt{c}\,\sigma \le \epsilon_1 \le \sqrt{c_F}\,\sigma$.

While the bounds in (3.23) are tighter than those of (3.17) and (3.20), we can simplify the r.h.s. of (3.23) for large values of mn. Specifically, when mn > 30, a simple (but good) approximation of the sample variance of e_{ij} [16;pp.343] yields

(3.24) $S_F^2/mn = (1/mn)\ [\sum_{i=1}^{m} \sum_{j=1}^{n} |e_{ij}|^2] \cong \sigma^2$.

By using (3.24), (3.23) becomes

Third Bounds (Modified)

(3.25) $\sqrt{c}\,\sigma \le \epsilon_1 \le \sqrt{mn}\,\sigma$.

Example 3.

Let A be a 7 x 7 matrix of rank 4 given by

(3.26) $$A = \begin{bmatrix} 3 & 2 & 1 & 7 & 4 & 5 & 3 \\ 1 & 4 & 2 & 6 & 5 & 10 & 3 \\ 8 & 1 & 5 & 13 & 5 & 7 & 0 \\ 4 & 2 & 7 & 15 & 11 & 11 & 4 \\ 1 & 2 & 1 & 3 & 2 & 5 & 1 \\ 2 & 1 & 3 & 5 & 3 & 5 & 0 \\ 3 & 10 & 1 & 5 & 2 & 21 & 1 \end{bmatrix}.$$

The SV's of A are given by $\{39.8, 15.99, 6.23, 3.00, 1.09 \times 10^{-15}, 5.63 \times 10^{-16}, 6.89 \times 10^{-17}\}$. For a level of confidence of $\alpha = 0.05$ and from (3.16) (3.17), (3.20), and (3.25), the upper and lower bounds of ϵ_1 (denoted by ϵ_U and ϵ_L, respectively), are given by

(3.27) First Bounds $\epsilon_L = 2\sigma \le \epsilon_1 \le 14\,\sigma = \epsilon_U$,

(3.28) Second Bounds $\epsilon_L = 3.75\,\epsilon \le \epsilon_1 \le 9.92\,\sigma = \epsilon_U$,

(3.29) Third Bounds (Modified) $\epsilon_L = 3.75\,\sigma \le \epsilon_1 \le 7\sigma = \epsilon_U$.

For noise variance σ^2 of 0.1, 0.01, and 0.001, Table 1 shows the six threshold bounds considered in (3.27)-(3.29). For later comparisons, the 3σ values are also tabulated.

σ^2	0.1		0.01		0.001	
Bounds	ϵ_L	ϵ_U	ϵ_L	ϵ_U	ϵ_L	ϵ_U
First	0.632	4.42	0.2	1.4	0.06	0.442
Second	1.185	3.137	0.375	0.99	0.118	0.313
Mod. Third	1.185	2.213	0.375	0.7	0.118	0.2213
3σ	0.948		0.3		0.0948	

Table 1. Thresholds Bounds of Example 3

Now, consider the observed matrix B=A+E, where $E = \{e_{ij}\}$ is a noise matrix with i.i.d. Gaussian r.v. of zero mean and variance σ^2. We evaluate the SV's $\{\beta_1, \beta_2, \ldots, \beta_7\}$ for different simulated runs of the noise. Table 2 shows the values of β_4 and β_5 for different values of the noise variance under different simulation runs.

RUN	$\sigma^2=0.1$		$\sigma^2=0.01$		$\sigma^2=0.001$	
	β_4	β_5	β_4	β_5	β_4	β_5
1	3.072	0.406	2.961	0.136	3.035	0.069
2	2.918	0.947	2.933	0.162	3.00	0.063
3	3.147	0.390	3.09	0.349	3.039	0.057
4	3.199	0.914	2.884	0.247	2.997	0.067
5	2.813	0.834	2.764	0.242	3.096	0.084
6	2.903	0.614	3.117	0.199	2.996	0.047
7	3.279	0.873	3.150	0.231	3.03	0.075

Table 2. Singular Values β_4 and β_5 of Ex. 3 under Different Simulation Runs

Of course, the main goal in this example is to find meaningful estimate $\hat{\epsilon}_1$ such that, $\beta_4 \geq \hat{\epsilon}_1 \geq \beta_5$ and thus determine the effective rank of B as being 4. As can be seen from Tables 1 and 2, by taking $\hat{\epsilon}_1$ to be any values in the interval $[\epsilon_L, \epsilon_U]$ obtained under the Modified Third Bounds, (3.25), we obtain the correct effective order under all three noise conditions. However, under the First or Second Bounds, although $\hat{\epsilon}_1 = \epsilon_U$ is still adequate to predict the correct effective rank for small noise variances of $\sigma^2 = 0.001$ and 0.01, it is not adequate under a larger noise variance of $\sigma^2 = 0.1$. Indeed, in the later cases, $\alpha_4 = 3 < 2\epsilon_U$ (which is equal to 8.84 under the First Bounds and 6.274 under the Second Bounds), and thus the sufficiency condition of Theorem 3 is not satisfied. We note a threshold of $\hat{\epsilon}_1 \doteq 3\sigma$, would result in correct decision in most but not all cases in this example.

The above discussions showed that the threshold bounds derived earlier for δ_1 under criterion 1 of (1.7) are relevant and useful in effective rank determination. Now, consider the use of criteria 2-5 for this example. For brevity, consider a typical simulated sequence of SV's of B given by (39.936, 15.558, 5.935, 3.072, 0.406, 0.3062, 0.0236} with $\sigma^2 = 0.1$. We note, β_4 and β_5 under Run 1 of Table 2 are from this set of SV's. Under criterion 2 of (1.8), from $\{\beta_t/\beta_1,\ t=1, \ldots, 7\}$, we obtain {1, 0.389, 0.148, 0.0769, 0.0102, 0.00767, 0.000590}. Since an explicit evaluation of δ_2 is unknown, the use of this set of data for rank determination is unclear. Under criterion 3 of (1.6), from $\{\beta_t/\beta_{t+1},\ t=1, \ldots, 6\}$, we obtain {2.560, 2.621, 1.931, 7.560, 1.326, 12.937}. Indeed, since $\beta_6/\beta_7 = 12.937 > \beta_4/\beta_5 = 7.560$, it is easy to conclude an effective rank of 6 instead of 4. Under criterion 4 of (1.10), $\{\sum_{i=t+1}^{7} \beta_i^2,\ t=0, \ldots 6\}$ yields $\{1881.856, 286.972, 44.921, 9.696, 0.259, 0.0943, 5.570 \times 10^{-4}\}$. Once again, without an explicit δ_4, rank determination from this set of data is difficult. Finally, under criterion 5 of (1.11), $\{(\sum_{i=1}^{t} \beta_i^2 / \sum_{i=1}^{7} \beta_i^2,\ t=1, \ldots, 7\}$ yields {0.920, 0.987, 9.997, 0.99993, 0.99997, 0.99999, 1}. If we use a $\delta_5 = 0.99$ as considered in [11], an effective rank of 3 is predicted. In general, the practical applicability of criteria 2-5 for effective rank determination without explicit knowledge of δ_i is not clear.

C. Column I.I.D. Random Model

In the development of the three bounds as well as in Example 3 in the last sub-section, we assumed that $\{e_{ij}\}$ is modelled as i.i.d. Gaussian r.v.'s. Consequently, the Third Bounds (as well as the Modified Third Bounds when applicable) are always tighter than the Second Bounds, which in turn are tighter than the First Bounds, as shown by (2.4) of Theorem 1. Now, suppose $\{e_{ij}\}$ is modeled only as column i.i.d., Gaussian r.v.'s. That is, for any j, $\{e_{ij},\ i=1, \ldots, m\}$ is assumed to be i.i.d. Gaussian r.v.'s of zero mean and variance σ^2. Thus, while the elements in any column are mutually independent, elements among different columns are not required to be mutually independent.

Upon a reinspection of the steps used in the derivation of the above bounds, it is clear that the First Bounds of (3.17) and the Second Bounds of (3.20) are still valid. However, the Third Bounds (and the Modified Third Bounds) are no longer valid since mutually independence of e_{ij} among different columns were used. In the next section the concept of column i.i.d. and the applicable Second Bounds of (3.20) are used.

IV. Applications to System Modeling.

In many communication, control, and signal processing problems, there is a fundamental interest in characterizing the effective order of the system. For example, the determination of the effective order of a communication channel is crucially needed in the design and analysis of equalizers and filters, in terms of arithmetic operations and complexity. In this section, we will consider the application of our previously defined threshold bounds for the efficient determination of the order of a linear system transfer function in the presence of observation noise.

Consider the autoregressive (AR) model

(4.1) $x(n) = a_1x(n-1)+a_2x(n-2)+\cdots+a_px(n-p) + u(n)$,

and the observed data

(4.2) $y(n) = x(n)+w(n)$.

$u(n)$ is an input $\{\pm A\}$ i.i.d. binary sequence, and $w(n)$ is a noise i.i.d. sequence uncorrelated with $u(n)$, with zero mean and variance σ_w^2.

In order to evaluate the order p of this A.R. model from the observed data $\{y(n)\}$, consider the $J\times(L+1)$ matrix A given by

(4.3) $A=[r(M), r(M-1), \ldots, r(M-L)]$,

where

(4.4) $r(M) = [R_y(M),R_y(M-1),\ldots, R_y(M-J+1)]^T$,

and $R_y(k)=E[y(n)y(n-k)]$ satisfy

(4.5) $J>L+1>p$, $M-L-J+1>0$.

From [13] and [15], it is known that matrix A has rank p, when (4.5) is satisfied. In practice, the system autocorrelation coefficients are unknown, and samples estimates $R'_y(k)$ have to be used instead. Denote

$$(4.6)\quad R'_y(k)=(1/N)\sum_{n-1}^{n=k} y(n)y(n-k),$$

where N is the number of available data.

Let B denote the same matrix as A of (4.3), but with all autocorrelation values $R_y(\cdot)$ replaced by $R'_y(\cdot)$ of (4.6). Because of the inherent uncertainty of these estimates, the last L+1-p singular values of B will not be exactly zero. Again we define the rank of B (and the order of the system) as equal to r, if for a given threshold ϵ_1, $\beta_r > \epsilon_1 \geq \beta_{r+1}$. From a statistical analysis of the sample autocorrelation estimates, the previous bounds for ϵ_1 are given by

$$(4.7)\quad \sqrt{c}\ \sigma \leq \epsilon_1 \leq \sqrt{(L+1)c}\ \sigma,$$

where c can be found from tables of Chi-square distributions with J degrees of freedom and a given level of significance, and $\sigma^2 \simeq (1/N)R_u(0)$ [15]. We note that, because of the symmetric structure of A in (4.3) (and thus also of B), any two consecutive Jx1 columns of B have (J-1) common elements. Thus, as discussed in subsection C of the last section, the tighter Third Bounds of (3.23) cannot be applied here, but the Second Bounds of (3.20) are used in (4.7). In addition to the bounds in (4.7), simulation results seem to indicate that an approximate threshold of

$$(4.8)\quad \tilde{\epsilon}_1 = k\sigma \cong k[(1/N)R_u(o)]^{1/2},$$

with typical values of k ranging between 3 and 4, yields the correct rank in a large number of cases of interest.

Example 4.

Consider the AR model with p=2 and coefficients a_1=0.4, a_2=-0.2, σ_w^2 = 0.1, and $R_u(o)$=1. For N=1000, M=7, J=4, and L=3, the 4 x 4 matrix B yields S.V.'s of (0.4756, 0.20109, 0.0685, and 0.029165.) Let $\sigma^2=0.001$, then from [16;p.559] with a level of significance $\alpha=0.05$ and for J=4 degrees of freedom, c=9.488. Thus the bounds of (4.7) yield

(4.9) $\epsilon_L=0.0974 \le \epsilon_1 \le 0.1948 = \epsilon_U$

By taking $\hat{\epsilon}_1$ to have any value in $[\epsilon_L, \epsilon_U]$ of (4.9), the order of the system can be predicted correctly as 2. From (4.8), the approximate threshold of $\tilde{\epsilon}_1 = 3\sigma$ yields $\tilde{\epsilon}_1 = 0.09486$ which is also sufficient to predict the correct order for this example.

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Appendix A

Proof of Theorem 1. The five inequalities in (2.4) can be shown easily proceeding from left to right.

i) Let a_{kl} be the element in A with the maximum absolute value. Then

$$\max |a_{ij}| = |a_{kl}| \le [\sum_{i=1}^{m} |a_{il}|^2]^{1/2} = \|a_l\|_2 \le \max \|a_j\|_2.$$

ii) Denote by $e_j = (0, 0, \ldots, 1, 0, \ldots, 0)^T$ an n x 1 vector with a 1 in the jth position and zero elsewhere. Since $\|e_j\|_2=1$ and $\|Ae_j\|_2=\|a_j\|_2$, then

$$\|a_j\|_2=(\|Ae_j\|_2/\|e_j\|_2 \le \max_{x\neq 0} (\|Ax\|_2/\|x\|_2) = \|A\|_2.$$

iii) From (2.1) and (2.2),

$$\|A\|_2 = \alpha_1 \le (\alpha_1^2 + \ldots + \alpha_k^2)^{1/2} = \|A\|_F.$$

iv) From (2.1),

$$\|A\|_F = [\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2]^{1/2} = [\sum_{j=1}^{n}\|a_j\|_2^2]^{1/2} \le \sqrt{n} \max \|a_j\|_2.$$

v) Finally,

$$\sqrt{n} \max \|a_j\|_2 = \sqrt{n} \max [\sum_{i=1}^{m} |a_{ij}|^2]^{1/2} \le \sqrt{mn} \max |a_{ij}|.$$

Appendix B

Proof of Theorem 3. From (2.5) of Lemma 2, we have

$\beta_{r+1} \le \epsilon_1$ and $|\beta_r-\alpha_r| \le \epsilon_1$. Suppose $\beta_r \le \alpha_r$.

Then $a_r - \beta_r \le \epsilon_1$ and $\beta_r \ge a_r - \epsilon_1 > 2\epsilon_1 - \epsilon_1 = \epsilon_1 \ge \beta_{r+1}$.

Otherwise, suppose $\beta_r > \alpha_r$. Then $\beta_r > \alpha_r > 2\epsilon_1 > \epsilon_1 \ge \beta_{r+1}$.